Option Pricing Under Stochastic Volatility with Incomplete Information

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Abstract

Options are analyzed and valued in the context of Merton's (1987) "Simple Model of Capital Market Equilibrium with Incomplete Information". We show now the derivation of the partial differential equation for options in the presence of shadow qcosts of incomplete information and stochastic volatility. We illustrate our approach by specific applications and show the dependancy of the option price on information and stochastic volatility. Then, we introduce information costs in a general diffusion model for asset prices which allows the description of stochastic volatility in an incomplete market. As in Norbert, Platen and Schweizer (1992), we show that the investor's choice of the minimal equivalent martingale measure is not changing, but the process of the price of the asset depends on incomplete information.

1 Introduction

Volatility is an important parameter in option pricing theory. Black and Scholes (1973)proposed an option valuation equation under the assumption of a constant volatility in a complete market without frictions. Engle (1982) developed a discrete-time model, to show that the volatility depends on its previous values.

The stochastic volatility problem has been examined by several authors. For example, Hull and White (1987), Wiggins (1987), Johnson and Shanno (1987) studied the general case in which the instantaneous variance of the stock price follows some geometric process. Scott (1989) and Stein and Stein (1991) used an arithmetic volatility in the study of option pricing. All these models describe (with precision) the effects of the volatility on the options prices. Stein and Stein (1991) and Heston (1993) proposed a dynamic approach for the volatility which is represented by an Ornstein-Ulhenbeck. It is difficult to find an analytic solution for the stochastic volatility option pricing problem.

Merton (1987) proposed a capital asset pricing model in the presence of the shadow costs of incomplete information. Bellalah (1990) applied the Merton (1987) model to the valuation of options under incomplete information. Bellalah and Jacquillat (1995) and Bellalah (1999) re derived the Black and Scholes (1973) equation in the context of Merton (1987) model, they obtained another version of the Black and Scholes equation within information uncertainty.

In this paper, we propose a general context for the pricing of options under stochastic volatility and information costs.

The first section provides a general concept for the valuation of options with shadow costs when the volatility is random. The second

Section examines some applications to several known models. The third section investigates a general process of a compatible asset with incompleteness in the market under information uncertainty and stochastic volatility.

2 Valuation of Options In the Presence of a Stochastic Volatility and Shadow Costs of Incomplete Information

2.1 The valuation model

The pricing of derivative securities in the presence of a random volatility needs the use of two processes : one for the underlying asset and one for the volatility. Consider the following dynamics for the underlying asset

$$dS = \mu S dt + \sigma S dW_1$$

and the following process for the volatility

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2$$

The two processes $dW_1 dW_2$ are Brownian-motions with a correlation coefficient ρ. The functions $p(S, \sigma, t)$ and $q(S, \sigma, t)$ are specified in a way that fits the dynamics of the volatility over time. Hence, the derivative asset price $V(S, \sigma, t)$ can be expressed as a function of the dynamics of the underlying asset price S, the volatility σ and time t. Since the volatility is not a traded asset, a problem arises because this new source of randomness can not be easily hedged away. The pricing of options in this context needs the search for two hedging contracts. The first is the underlying asset. The second can be an option that allows a hedge against volatility risk. Following the same logic as in the original Black-Scholes model (1973), consider a portfolio comprising a long position in the option V, a short position of Δ units of the underlying asset and a short position of $-\Delta_1$ units of an other option with value $V_1(S, \sigma, t)$:

$$\Pi = V - \Delta S - \Delta_1 V_1 \tag{1}$$

Over a short interval of time dt, applying Ito's lemma for the functions S, σ and t gives the change in the value of this portfolio as:

$$\begin{aligned} d\Pi = &\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt \\ &- \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt \\ &+ \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S}\Delta\right) dS + \left(\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma}\right) d\sigma \end{aligned}$$

All the sources of randomness in the portfolio value resulting from dS can be eliminated by setting the quantity before dS equal to zero, or

$$\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0$$

and also by setting the quantity before $d\sigma$ equal to zero, or

$$\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0$$

After eliminating the stochastic terms, the terms in dt must yield the deterministic return as in a Black-Scholes "hedge" portfolio. Hence, the instantaneous return on the portfolio must be the risk-free rate plus information costs on each asset in the portfolio as in Bellalah (1999). This gives

$$\begin{aligned} d\Pi = &\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + \rho\sigma S q \frac{\partial^2 V}{\partial S \partial \sigma}dt + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial \sigma^2}dt \\ &- \Delta_1 \left(\frac{\partial V_1}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2}dt + \rho\sigma S q \frac{\partial^2 V_1}{\partial S \partial \sigma}dt + \frac{1}{2}q^2 \frac{\partial^2 V_1}{\partial \sigma^2}dt\right) \\ &= [(r + \lambda_V)V - (r + \lambda_S)\Delta S - (r + \lambda_{V_1})\Delta_1 V_1]dt, \end{aligned}$$

Isolating the terms in V and V_1 gives

$$\frac{\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + \rho\sigma S q \dfrac{\partial^2 V}{\partial S \partial \sigma} + \dfrac{1}{2}q^2 \dfrac{\partial^2 V}{\partial \sigma^2} + (r + \lambda_S)S\dfrac{\partial V}{\partial S} - (r + \lambda_V)V}{\dfrac{\partial V}{\partial \sigma}}$$

$$= \frac{\dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + \rho\sigma S q \dfrac{\partial^2 V_1}{\partial S \partial \sigma} + \dfrac{1}{2}q^2 \dfrac{\partial^2 V_1}{\partial \sigma^2} + (r + \lambda_S)S\dfrac{\partial V_1}{\partial S} - (r + \lambda_{V_1})V_1}{\dfrac{\partial V_1}{\partial \sigma}}$$

Since the two options differ by their strikes, payoffs and maturities, this implies that both sides of the equation are independent of the contract type. Since both sides are functions of the independent variables S, σ and t, we have

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S q \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial \sigma^2} + (r + \lambda_S)S\frac{\partial V}{\partial S} - (r + \lambda_V)V \\ &= -(p - \delta q)\frac{\partial V}{\partial \sigma}, \end{aligned}$$

for a function $\delta(S, \sigma, t)$ referred to as the market price for risk or volatility risk. This equation can also be written as

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S q \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial \sigma^2} + (r + \lambda_S)S\frac{\partial V}{\partial S} \\ &+ (p - \delta q)\frac{\partial V}{\partial \sigma} - (r + \lambda_V)V = 0 \end{aligned} \tag{2}$$

This equation shows two hedge ratios $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial \sigma}$. The term $(p - \delta q)$ is known as the risk-neutral drift rate.

2.2 Market price of volatility risk

Suppose the investor holds only the option V which is hedged only by the underlying asset S in the following portfolio

$$\Pi = V - \Delta S$$

Over a short interval of time dt, the change in the value of this portfolio can be written as

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \frac{\partial V}{\partial \sigma} d\sigma$$

In the standard delta-hedging, the coefficient of dS is zero and we have

$$\begin{aligned} d\Pi - [(r + \lambda_V)V - (r + \lambda_S)\Delta S]dt &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma}\right. \\ &\quad + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + (r + \lambda_S) S \frac{\partial V}{\partial S} \\ &\quad \left. - (r + \lambda_V)V\right] dt + \frac{\partial V}{\partial \sigma} d\sigma \\ &= q \frac{\partial V}{\partial \sigma}(\delta dt + dW_2) \end{aligned}$$

This results from equations (1) and (2). The term dW_2 represents a unit of volatility risk. There are δ units of extra-return, given by dt for each unit of volatility risk.

2.3 The market price of risk for traded assets

In the Black-Scholes analysis, the hedging portfolio is constructed using the option and its underlying tradable asset. Consider the construction of a portfolio as before using two options V and V_1 with different characteristics, the initial portfolio value would be

$$\Pi = V - \Delta_1 V_1$$

Note that there are none of the underlying asset in this portfolio. Using the same methodology as before gives the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\mu - \delta_S \sigma) S \frac{\partial V}{\partial S} - (r + \lambda_V)V = 0 \tag{3}$$

The variable asset S is the value of a traded asset. Then $V = S$ must be a solution to this last equation. Substituting $V = S$ in the last equation gives

$$(\mu - \delta_S \sigma)S - (r + \lambda_S)S = 0$$

The market price of risk for a traded asset in the presence of information costs

$$\delta_S = \frac{\mu - (r + \lambda_S)}{\sigma}$$

Substituting δ_S in (3) gives the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + \lambda_S) S \frac{\partial V}{\partial S} - (r + \lambda_V)V = 0$$

This is the Black-Scholes equation in the presence of information costs.

3 Generalization of Certain Model with Stochastic Volatility and Information Costs

3.1 Generalization of the Hull and White 1987 model

We consider the following model

$$\begin{aligned} dB_t &= (r + \lambda_B) B_t dt \\ dS_t &= \mu(S_t, \sigma_t, t) S_t dt + \sigma_t S_t dW_t^1 \\ d\nu_t &= \gamma(\sigma_t, t)\nu_t dt + \delta(\sigma_t, t)\nu_t dW_t^2 \end{aligned} \tag{4}$$

where S_t denotes the stock price at time t, $\nu_t = \sigma_t^2$ its instantaneous variance, and r the riskless interest rate, which is assumed to be constant. W^1 and W^2 are Brownian motions under P, they are independent. ν_t has no systematic risk. This yields a unique option price which can be computed as the (conditional) expectation of the discounted terminal payoff under a risk-neutral probability measure $\tilde{P}$. Put differently, $\tilde{P}$ is obtained from P by means of a Girsanov transformation such that

$$\begin{aligned} dB_t &= (r + \lambda_B) B_t dt \\ dS_t &= (r + \lambda_S) S_t dt + \sigma_t S_t d\tilde{W}_t^1 \\ d\nu_t &= \gamma(\sigma_t, t)\nu_t dt + \delta(\sigma_t, t)\nu_t d\tilde{W}_t^2 \end{aligned} \tag{5}$$

under $\tilde{P}$, where $\tilde{W}^1$, $\tilde{W}^2$ are independent Brownian motions under $\tilde{P}$. The risk-neutral dynamics of the bond and the underlying asset are used in Bellalah (1999). The portfolio value would be

$$\Pi = V - \Delta S - \Delta' B_t$$

with Δ' units of the bond. When we apply the methodology of the previous section to this model, equation (2) gives:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\nu_t S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \rho\sigma_t^3 S_t \xi \frac{\partial^2 V}{\partial S \partial \nu_t} + \xi^2 \nu_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial \nu_t^2} \\ &\quad + (r + \lambda_S) S_t \frac{\partial V}{\partial S_t} + (\gamma - \delta\xi)\nu_t \frac{\partial V}{\partial \nu_t} + (r + \lambda_B) B_t \frac{\partial V}{\partial B_t} - (r + \lambda_V)V = 0 \end{aligned} \tag{6}$$

with $\tilde{W}^1, \tilde{W}^2$ independent Brownian motion under the probability $\tilde{P}$ ($\rho = 0$)and λ_S is the information cost of the security S_t. The investor paid the shadow cost λ_S if he does not know the asset. Also λ_B is the information cost of the bond B_t and it is equal to zero if the asset is reskless. We suppose that $\delta = 0$, The option price is then given by

$$V(t, S_t) = \tilde{E}\left[\frac{B_t}{B_T}(S_T - K)^+ |\mathcal{F}_t\right] = e^{-(r+\lambda_B)(T-t)}\tilde{E}[(S_T - K)^+ |\mathcal{F}_t] \quad (7)$$

To obtain a more specific form for V, we use the additional assumption contained in (5) and the independence of W^1 , W^2 that the instantaneous variance v is not influenced by the stock price S. Setting

$$\overline{v}_{t,T} = \frac{1}{T-t}\int_t^T v_s ds \quad (8)$$

They show that the conditional distribution of $\frac{S_T}{S_t}$ under $\tilde{P}$, given $\overline{v}_{t,T}$, is log-normal with parameters $(r + \lambda_B)(T - t)$ and $\overline{v}_{T-t}$. This allows to write V as

$$V(t, S_t, \sigma_t^2) = \int_0^\infty u_{BS}(t, S_t, \overline{v}_{t,T})dF(\overline{v}_{t,T}|S_t, \sigma_t^2) \quad (9)$$

where V_{BS} denotes the usual Black-Scholes (1973) price corresponding to the variance $\overline{v}_{t,T}$ and F is the conditional distribution under $\tilde{P}$ of $\overline{v}_{t,T}$ given S_t and σ_t^2. This is equivalent to write :

$$V(t, S_t, \sigma_t^2) = \int_0^\infty V_{BS}(t, S_t, \overline{v}_{t,T})h(\overline{v}_{t,T}|S_t, \sigma_t^2)d\overline{v}_{t,T} \quad (10)$$

with

$$V_{BS}(\overline{v}) = S_t N(d_1) - Xe^{-(r+\lambda_S)(T-t)}N(d_2)$$

and $d_1 = \dfrac{\log(S_t/K) + (r + \lambda_S + \overline{v}/2)(T - t)}{\sqrt{\overline{v}(T - t)}}, \quad d_2 = d_1 - \sqrt{\overline{v}(T - t)}$.

When $\mu = 0$ and as in H and White (1987) we have:

$$\begin{aligned} V(S, \sigma_t^2) = V_{BS}(\overline{v}) &+ \frac{1}{2}\frac{S\sqrt{T-t}N'(d_1)(d_1d_2 - 1)}{4\sigma^3} \\ &\times \left[\frac{2\sigma^4(e^k - k - 1)}{k^2} - \sigma^4\right] \\ &+ \frac{1}{6}\frac{S\sqrt{T-t}N'(d_1)[(d_1d_2 - 1)(d_1d_2 - 3) - (d_1^2 + d_2^2)]}{8\sigma^5} \\ &\times \sigma^6\left[\frac{e^{3k} - (9 + 18k)e^k + (8 + 24k + 18k^2 + 6k^3)}{3k^3}\right] + \ldots, \end{aligned} \quad (11)$$

avec $k = \xi^2(T - t), \quad N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

3.2 Generalization of Wiggins's model

Under the assumption of the continuous trading, without frictions, in a complete market, Wiggins (1987) use the following dynamics for the asset and the volatility:

$$\begin{aligned} dS_t &= \mu(S_t, \sigma_t, t)S_t dt + \sigma_t S_t dW_{S_t} \\ d\sigma_t &= f(\sigma_t)dt + \theta\sigma_t dW_{\sigma_t} \end{aligned} \quad (12)$$

with dW_{S_t}, dW_{σ_t} are processes of Wiener, the correlation coefficient between stock returns and volatility movements is $\rho dt = dW_{S_t}dW_{\sigma_t}$ and $(dP/P)(dS_t/S_t) = 0$. The instantaneous rate of return on the hedge portfolio P is

$$dP/P = wdV/V + (1 - w)dS_t/S_t$$

with w the fraction invested in the contingent claim V and $(1 - w)$ the fraction invested in the stock S. Equation (2) is equivalent in this case to:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + \rho\sigma_t^2\theta S_t\frac{\partial^2 V}{\partial S_t\partial\sigma_t} + \theta^2\sigma_t^2\frac{1}{2}\frac{\partial^2 V}{\partial\sigma_t^2} \\ &\quad + (r + \lambda_S)S_t\frac{\partial V}{\partial S_t} + (f(\sigma_t) - \delta\theta\sigma_t)\frac{\partial V}{\partial\sigma_t} - (r + \lambda_V)V = 0 \end{aligned} \quad (13)$$

As in Wiggins (1987), we can write the following equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + \rho\sigma_t^2\theta S_t\frac{\partial^2 V}{\partial S\partial\sigma_t} + \theta^2\sigma_t^2\frac{1}{2}\frac{\partial^2 V}{\partial\sigma_t^2} + (r + \lambda_S)S_t\frac{\partial V}{\partial S_t} \\ &\quad + \left[f(\sigma_t) - (\mu - r - \lambda_S)\rho\theta + \Phi(.)\theta\sigma_t\sqrt{(1 - \rho^2)}\right]\frac{\partial V}{\partial\sigma_t} - (r + \lambda_V)V = 0 \end{aligned} \quad (14)$$

We conclude that the market price of risk affects the term given by Wiggins (1987)$\Phi(.) = (\mu_P - r - \lambda_P)/\sigma_P$. This term is the expected excess return per unit risk, or the market price of risk, for the hedge portfolio. It represents the return-to-risk tradeoff required by investors for bearing the volatility risk of the stock.

$$\Phi(.) = \frac{\delta\sigma_t - (\mu - r - \lambda_S)\rho}{\sigma_t\sqrt{(1 - \rho^2)}} \quad (15)$$

The market price of risk depends on the information cost of the stock and the stochastic volatility.

3.3 Generalization of Stein and Stein's model

In this model, the stock price dynamics are given by the following process:

$$dS_t = \mu(S_t, \sigma_t, t)S_t dt + \sigma_t S_t dW_1$$

The volatility follows an Ornstein-Uhlenbeck process:

$$d\sigma_t = \varpi(\sigma_t - \theta)dt + kdW_2$$

The Weiner processes dW_1, dW_2 are uncorrelated. When equation (2) is applied in this context, we have:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} + k^2\frac{1}{2}\frac{\partial^2 V}{\partial\sigma_t^2} \\ &\quad + (r + \lambda_{S_t})S_t\frac{\partial V}{\partial S_t} + [-\varpi(\sigma_t - \theta) - \delta k]\frac{\partial V}{\partial\sigma_t} - (r + \lambda_V)V = 0 \end{aligned} \quad (16)$$

When $\delta = 0$ or to be a constant, equation (16) has a solution with the same form as in Stein and Stein (1991). The solution depends on information costs of V and the underlying asset S. The option price has the following form:

$$V = e^{-(r+\lambda_V)} \int_{S_t=K}^{\infty} [S_t - K] H(S_t, t \mid \varpi, r + \lambda_{S_t}, k, \theta) dS_t \tag{17}$$

with $H(S_t, t)$ is the price distribution of the underlying asset at the time t with a non-zero drift of S_t.

3.4 Generalization of Heston's model

The underlying asset and the volatility follow the diffusion process:

$$\begin{aligned} dS_t &= \mu(S_t, \sigma_t, t) S_t dt + \sqrt{\nu_t} S_t dW_t^1 \\ d\nu_t &= \kappa(\theta - \nu_t) dt + \sigma_t \sqrt{\nu_t} dW_t^2 \end{aligned} \tag{18}$$

with ρ the correlation coefficient between dW_t^1, dW_t^2.

In this case, the value of any option $V(S_t, \nu_t, t)$ must satisfy the following partial differential equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \rho \sigma_t \nu_t S_t \frac{\partial^2 V}{\partial S_t \partial \nu_t} + \frac{1}{2} \nu_t S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \sigma_t^2 \nu_t \frac{1}{2} \frac{\partial^2 V}{\partial \nu_t^2} \\ &+ (r + \lambda_S) S_t \frac{\partial V}{\partial S_t} + [\kappa(\theta - \nu_t) - \delta \sigma_t \sqrt{\nu_t}] \frac{\partial V}{\partial \nu_t} - (r + \lambda_V) V = 0 \end{aligned} \tag{19}$$

Under the same assumption as in Heston (1993), it is possible to obtain solution to equation (19). This solution depends on information costs λ_S. In fact, an European call with a strike price K and maturing at time T, satisfies the equation (19) subject to the following boundary conditions

$$\begin{aligned} V(S, \nu_t, t) &= Max(o, S - K) \\ V(0, \nu_t, t) &= 0 \\ \frac{\partial V}{\partial S_t}(\infty, \nu_t, t) &= 1 \\ (r + \lambda_S) S_t \frac{\partial V}{\partial S_t} + \kappa(\theta) \frac{\partial V}{\partial \nu_t} - (r + \lambda_V) V + \frac{\partial V}{\partial t} &= 0 \\ V(S, \infty, t) &= S \end{aligned}$$

By analogy with the Black et Scholes (1973) formula, Heston (1993) gives a solution of the form

$$V(S, \nu_t, t) = SP_1 - KP(t, T) P_2 \tag{20}$$

with $P(t, t + \tau) = e^{-(r+\lambda_S)\tau}$ the price at time t of a unit discount bond that matures at time $t + \tau$. The first term of the right side of the solution $V(S, \nu_t, t)$ is the present value of the underlying asset upon optimal exercise. The second term is the present value of the strike-price. Both of these terms must satisfy the equation. It is convenient to write them in terms of the logarithm, $(19) x = \ln(S)$. By substitution of the solution in equation (19), $(20) P_1$ and P_2 must satisfy the following equation:

$$\begin{aligned} &\frac{\partial P_j}{\partial t} + \frac{1}{2} \nu_t \frac{\partial^2 V}{\partial x} + \rho \sigma_t \nu_t S_t \frac{\partial^2 V}{\partial S_t \partial \nu_t} + \sigma_t^2 \nu_t \frac{1}{2} \frac{\partial^2 P_j}{\partial \nu_t^2} \\ &+ (r + \lambda_S + u_j \nu_t) \frac{\partial P_j}{\partial x} + (a - b_j \sqrt{\nu_t}) \frac{\partial P_j}{\partial \nu_t} = 0 \end{aligned} \tag{21}$$

for $j = 1, 2$ where $u_1 = 1/2$, $u_2 = -1/2$, $a = \kappa\theta$, $b_1 = (\kappa - \rho\sigma_t)\sqrt{\nu_t} + \delta\sigma_t$, $b_2 = \kappa\sqrt{\nu_t} + \delta\sigma_t$

Following the same resolution method in Heston (1993) for the equation (21), we obtain the solution of the characteristic function:

$$f_j(x, \nu_t, t; \phi) = exp[C(T - t; \phi) + D(T - t; \phi)\nu_t + ix\phi] \tag{22}$$

when

$$\begin{aligned} C(\tau; \phi) &= i(r + \lambda_{S_t})\phi\tau + \frac{a}{\sigma_t^2} \left\{ (b_j - i\rho\sigma_t\phi + d)\tau - 2 \ln \left[\frac{1 - ge^{d\tau}}{1 - g} \right] \right\} \\ D(\tau; \phi) &= \frac{b_j - i\rho\sigma_t\phi + d}{\sigma_t^2} \left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}} \right] \end{aligned}$$

and $g = \dfrac{b_j - i\rho\sigma_t\phi + d}{b_j - i\rho\sigma_t\phi - d}$, $\quad d = \sqrt{(i\rho\sigma_t\phi - b_j)^2 - \sigma_t^2(2iu_j\phi - \phi^2)}$

By inverting the characteristic functions f_j, we obtain the desired probabilities:

$$P_j(x, \nu_t, t; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re \left[\frac{(e^{-i\phi \ln K})^* f_j(x, \nu_t, T; \phi)}{i\phi} \right] d\phi \tag{23}$$

with $f_j(x, \nu_t, T; \phi) = e^{i\phi x}$.

3.5 Generalization of Johnson and Shanno's model

We consider the following model:

$$\begin{aligned} dS_t &= \mu_{S_t} S_t dt + \sigma_t S_t^{\alpha} dW_t^1 \\ d\sigma_t &= \mu_{\sigma_t} \sigma_t dt + \sigma_t \sigma_{\sigma_t} dW_t^2 \end{aligned} \tag{24}$$

with $dW_t^1 dW_t^2 = \rho dt$.

When equation (2) is applied to the model, we obtain:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_t^2 S_t^{2\alpha} \frac{\partial^2 V}{\partial S_t^2} + \rho \sigma_t^3 S_t^{\alpha} \sigma_{\sigma_t} \frac{\partial^2 V}{\partial S \partial \sigma_t} + \sigma_t^2 \sigma_{\sigma_t}^2 \frac{1}{2} \frac{\partial^2 V}{\partial \sigma_t^2} \\ &+ (r + \lambda_S) S_t \frac{\partial V}{\partial S_t} + (\mu_{\sigma_t} - \delta \sigma_{\sigma_t}) \sigma_t \frac{\partial V}{\partial \sigma_t} - (r + \lambda_V) V = 0 \end{aligned} \tag{25}$$

Johnson and Shanno (1987), suppose that the risk premium of the volatility is zero. Consequently, we have:

$$\delta = \frac{\mu_{\sigma_t}}{\sigma_{\sigma_t}}$$

3.6 A general Markovian model with shadow costs of incomplete information

In this subsection we present a general model in which we include information cost and volatilities stochastic. We consider the following multi-dimensional diffusion process:

$$dX_t^i = a^i(t, X_t)dt + \sum_{j=1}^{n} b^{ij}(t, X_t)W_t^j \tag{26}$$

for $i = 1, \ldots, m$, where

$$a^i, b^{ij} : [0, T] \times \mathbb{R}^{m+1} \to \mathbb{R}$$

are measurable functions. The process $W = (W^1, \ldots, W^n)$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, Q)$, and $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the Q-augmentation of the filtration generated by W. We assume that the coefficients a^i, b^{ij} satisfy appropriate growth and Lipschitz conditions so that the solution of (26) is a Markov process. We also remark that under suitable continuity and nondegeneracy conditions on the coefficients, $\mathbb{F}$ coincides with the natural filtration $\mathbb{F}^X$ of X. This model will be interpreted in the following way. The component X^0 describes the risk less asset; setting $B := X^0$, we shall take $b^{0j} \equiv 0$ for $j = 1, \ldots, n$ and $a^0(t, x) = (r(t, X_t) + \lambda_B)x^0$, so

$$dB_t = (r(t, X_t) + \lambda_B)B_t dt \tag{27}$$

We assume that

$$\int_0^T |r(s, X_s) + \lambda_X| ds \leq L < \infty \qquad Q - a.s. \tag{28}$$

for some $L > 0$ and $X = (B, X^1, \ldots X^m)$. According to the analysis of Merton (1987),we assume that $\lambda_X = \lambda(s, X_s)$ is measured in units of expected return. We shall work with only one stock. The component X^1 describes its price process and is denoted by S. The other components of X can then be used to model the additional structure of the market in which S is embedded. In this general framework, an option or contingent claim will be a random variable of the form $g(X_T)$. The classical example is provided by European call option with strike price K which corresponds to the claim $(S_T - K)^+$. Since the process X will usually contain more components than just the bond $B = X^0$ and the stock price $S = X^1$, claim can depend on many things other than just the terminal stock price S_T. In fact, the only serious restriction is that the underling process X (but not necessarily S) should be Markovian. This implies that (subject to some integrability conditions) we can associate to any contingent claim $g(X_T)$ an option pricing function

$$V : [0, T] \times \mathbb{R}^{m+1} \to \mathbb{R}$$

defined by

$$V(t, x) = E_Q\left[exp\left(-\int_t^T r(s, X_s) + \lambda_X ds\right) g(X_T^{t,x})\right] \tag{29}$$

where $(X_s^{t,x})_{t\leq s\leq T}$ denotes the solution of (26) starting from x at time t, i.e., with $X_t^{t,x} = x \in \mathbb{R}^{m+1}$. To illustrate the previous analysis, we give the following example:

$$\begin{aligned} dB_t &= (r(t, X_t) + \lambda_B)B_t dt \\ dS_t &= (r(t, X_t) + \lambda_S)S_t dt + \sigma_t S_t dW_t^1 \\ d\sigma_t &= -q(\sigma_t - \varsigma_t)dt + p\sigma_t dW_t^2 \\ d\varsigma_t &= \frac{1}{\alpha}(\sigma_t - \varsigma_t)dt \end{aligned} \tag{30}$$

with $p > 0, q > 0, \alpha > 0$ and dW_t^1, dW_t^2 are independent Brownian motion under the probability Q. The processes of σ, ς are respectively the instantaneous and weighted average volatility of the stock. The equation for σ shows that the instantaneous volatility σ_t is distributed by some external noise (with an intensity p) and at the same time continuously pulled back toward the average volatility ς_t. The parameter q measures the strength of this restoring force or speed of adjustment. The equation (2) becomes in this case as follows

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \rho\sigma^2 p S_t \frac{\partial^2 V}{\partial S \partial \sigma_t} + p^2 \sigma_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial \sigma_t^2} + (r + \lambda_S)S_t \frac{\partial V}{\partial S_t} \\ &+ (-q(\sigma_t - \varsigma_t) - \delta p \sigma_t)\frac{\partial V}{\partial \sigma_t} + (r + \lambda_B)B_t \frac{\partial V}{\partial B_t} + \frac{1}{\alpha}(\sigma_t - \varsigma_t)\frac{\partial V}{\partial \varsigma_t} \\ &- (r + \lambda_V)V = 0 \end{aligned} \tag{31}$$

Or

$$\frac{\partial V}{\partial \varsigma_t} = \frac{\partial V}{\partial \sigma_t}\frac{\partial \sigma_t}{\partial \varsigma_t}$$

The equation (31) becomes

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \rho\sigma^2 p S_t \frac{\partial^2 V}{\partial S \partial \sigma_t} + p^2 \sigma_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial \sigma_t^2} + (r + \lambda_S)S_t \frac{\partial V}{\partial S_t} \\ &+ (-q(\sigma_t - \varsigma_t) - \delta p \sigma_t + \frac{1}{\alpha}(\sigma_t - \varsigma_t)\frac{\partial \sigma_t}{\partial \varsigma_t})\frac{\partial V}{\partial \sigma_t} + (r + \lambda_B)B_t \frac{\partial V}{\partial B_t} \\ &- (r + \lambda_V)V = 0 \end{aligned} \tag{32}$$

This equation can be written as follows:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \rho\sigma^2 p S_t \frac{\partial^2 V}{\partial S \partial \sigma_t} + p^2 \sigma_t^2 \frac{1}{2}\frac{\partial^2 V}{\partial \sigma_t^2} + (r + \lambda_S)S_t \frac{\partial V}{\partial S_t} \\ &+ \left[\left(\frac{1}{\alpha}\frac{\partial \sigma_t}{\partial \varsigma_t} - q\right)(\sigma_t - \varsigma_t) - \delta p \sigma_t\right]\frac{\partial V}{\partial \sigma_t} + (r + \lambda_B)B_t \frac{\partial V}{\partial B_t} \\ &- (r + \lambda_V)V = 0 \end{aligned} \tag{33}$$

with λ_B, λ_S and λ_V indicate the information costs respectively for the bond, the stock and the option.

4 The Incomplete Market and the Minimal Equivalent Martingale Measure with Information Costs

We shall work with a model considerably more general than. It contains one risk less asset B and m risky assets (26)$S^i, i = 1, \ldots, m$. The bond price B and the stock prices S^i are given by the stochastic differential equation

$$\begin{aligned} dB_t &= (r + \lambda_B)B_t dt \\ dS_t^i &= \mu_t^i S_t^i dt + S_t^i \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \end{aligned} \tag{34}$$

Here, $W = (W^1, \ldots, W^n)*$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ denotes the P-augmentation of the filtration generated by W. We take $n \ge m$ so that there are at least as many sources of uncertainty as there are stocks available for trading. All processes will be defined on $[0, T]$, where the constant $T > 0$ denotes the terminal time for our problem. We assume that the interest rate $r = (r_t)_{0\le t\le T}$, the vector $\mu = (\mu_t)_{0\le t\le T} = (\mu_t^1, \ldots, \mu_t^m)_{0\le t\le T}$ of stock appreciation rates, the volatility matrix $\sigma = (\sigma_t)_{0\le t\le T} = (\sigma_t^{ij})_{0\le t\le T, i=1,\ldots,n, j=1,\ldots,m}$ and the vector $(\lambda_X)_{0\le t\le T} = (\lambda_B, \lambda_{S^1} .., \lambda_{S^m})_{0\le t\le T}$ of the assets'information costs , are progressively measurable with respect to $\mathbb{F}$. The interest rate r and λ_X satisfies

$$\int_0^T |r_u + \lambda_X| du \le L < \infty \quad P - a.s. \qquad \text{for some } L > 0$$

This implies that the bond price process B is bounded above and away from 0, uniformly in t and ω. We also assume that the matrix σ_t has full rank m for every t so that the matrix $(\sigma_t \sigma_t^*)^{-1}$is well defined. This means that the basic assets, namely the stock prices, have been chosen in such a way that they are all nonredundant. Consider a "small investor", i.e. , an economic agent whose actions do not influence prices, who trades in the stocks and the bond. His trading strategy can be described at any time t by his total wealth V_t and by the amounts π_t^i invested in the ith stock for $i = 1, \ldots, m$. The amount invested in the bond is then given by $V_t - \sum_{i=1}^m \pi_t^i$. We shall call $\pi = (\pi_t)_{0\le t\le T} = (\pi_t^1, \ldots, \pi_t^m)^*_{0\le t\le T}$ a portfolio process if π is progressively measurable with respect to $\mathbb{F}$ and satisfies

$$\int_0^T ||\sigma_u^* \pi_u||^2 du < \infty \qquad P - a.s.$$

and

$$\int_0^T |\pi_u^* (\mu_u - r_u 1 - \lambda_S| du < \infty \qquad P - a.s$$

where $1 = (1, \ldots, 1)^* \in \mathbb{R}^m$ and $\lambda_S = (\lambda_{S^1}, \ldots, \lambda_{S^m})$ the vector of shadow costs of the risky assets. The trading strategy is called self-financing if all changes in the wealth process are entirely due to gains or losses from trading in the stocks and bond. For such a strategy, we denote two predictable processes in $\mathbb{R}^{m+1}$ the first is $(\eta_t)_{0\le t\le T}$ the quantity of the riskless asset or the bond and the second one is $\xi_t = (\xi_t^1, \ldots, \xi_t^m)_{0\le t\le T}$ the quantity of the risky assets hold in such portfolio. the wealth process V must satisfy the following equation:

$$dV_t = \eta_t dB_t + \sum_{i=1}^{m} \xi_t^i dS_t^i \tag{35}$$

Substituting equation (34) in equation (35) and setting $\pi_t^i = \xi_t^i S_t^i$ we have:

$$\begin{aligned} dV_t &= \sum_{i=1}^{m} \pi_t^i \left(\mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right) \\ &\quad + (V_t - \sum_{i=1}^{m} \pi_t^i)(r_t + \lambda_B) dt \\ &\Longleftrightarrow \\ dV_t &= \sum_{i=1}^{m} \pi_t^i \left(\mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j \right) \\ &\quad + V_t(r_t + \lambda_V) - \sum_{i=1}^{m} \pi_t^i (r_t + \lambda_i) dt \end{aligned} \tag{36}$$

With no arbitrage, we have:

$$dV_t = \sum_{i=1}^{m} \pi_t^i (\mu_t^i - r_t - \lambda_i) dt + \sum_{i=1}^{m} \pi_t^i \sum_{j=1}^{n} \sigma_t^{i,j} dW_t^j + V_t(r_t + \lambda_V) dt$$

With λ_V is the vector of information costs $(\lambda_B, \lambda_1, \ldots, \lambda_m)$because the value of the option is equal to the value of the portfolio in this strategy. Equation (36) is equal to:

$$\begin{aligned} dV_t &= \left(\pi_t^* [\mu_t - r_t \mathbf{1} - \lambda_S] + (r_t + \lambda_V) V_t \right) dt \\ &\quad + \pi_t^* \sigma_t dW_t, \qquad 0 \le t \le T \end{aligned} \tag{37}$$

The discounted wealth process $V' = V/B$ is then given by

$$dV_t' = \pi_t'^* [\mu_t - r_t 1 - \lambda_S] dt + V_t' (\lambda_V - \lambda_B 1) + \pi_t'^* \sigma_t dW_t \tag{38}$$

$$\Longleftrightarrow$$

$$dV_t' = \pi_t'^* [\mu_t - r_t \mathbf{1} - \lambda_S] dt + V_t' (\lambda_S - \lambda_B \mathbf{1}) + \pi_t'^* \sigma_t dW_t \tag{39}$$

with $\pi_t'^* = \pi_t^* / B_t$. Thus, any portfolio, process π uniquely determines a wealth process V such that π and V together constitute a self-financing strategy. We remark that the process give by the equation (37) depend on the vector of the shadow costs λ_S [because that the value of the market price depend on λ_S] and on λ_B. If we interpret the process V as the price of some assets, we can remind the definition of any "general" asset:

Definition: A general asset is any asset whose value A is a semi martingale with respect to P and $\mathbb{F}$.

Remark: For the following study, the general asset A has the following form

$$dA_t = \upsilon_t^* dW_t + dF_t \qquad 0 \le t \le T \tag{40}$$

with F un $\mathbb{F}$-adapted process with paths of finite variation and the process $\upsilon = (\upsilon^1, \ldots, \upsilon^n)^*$ is progressively measurable with respect to $\mathbb{F}$ and satisfies:

$$\int_0^T ||\upsilon_u||^2 du < \infty \qquad P - a.s.$$

We recall also the concept of an equivalent martingale for S:

Definition 1: A probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ is called equivalent martingale measure for S if

i)- $\tilde{P}$ and P have the same null sets, $\tilde{P} \approx P$. In particular, this implies $\tilde{P} = P$ on $\mathcal{F}_0$.
ii)- The discounted price process $S' = S/B$ is a vector martingale under $\tilde{P}$.

Definition 2: An equivalent martingale measure $\hat{P}$ for S is called minimal if any local P-martingale, orthogonal to $S^i, i = 1, \ldots, m$ 0remains a local martingale under $\hat{P}$.

We begin by describing more precisely the equivalent martingale measures for S. If $\tilde{P}$ is any equivalent martingale measure for S and

$$\tilde{Z}_t = E_P\left[\frac{d\tilde{P}}{dP}\bigg|\mathcal{F}_t\right] = \frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_t}, \quad 0 \le t \le T \tag{4.5}$$

denotes a continuous version of the density process of $\tilde{P}$ with respect to P, then $\tilde{Z}$ can be written as

$$\tilde{Z}_t = exp\left(-\int_0^t \tilde{\gamma}_u^* dW_u - \frac{1}{2}\int_0^t ||\tilde{\gamma}_u||^2 du\right), \qquad 0 \le t \le T \tag{41}$$

where $\tilde{\gamma} = (\tilde{\gamma}^1, \ldots, \tilde{\gamma}^n)^*$ is adapted to $\mathbb{F}$ and satisfies the following condition

$$\int_0^T ||\tilde{\gamma}_u||^2 du < \infty \qquad P - a.s. \tag{42}$$

and

$$\sigma_t \tilde{\gamma}_t = [\mu_t - r_t 1 - \lambda_S], \qquad 0 \le t \le T \tag{43}$$

if we suppose the existence and the uniqueness of the minimal equivalent martingale measure[1] $\hat{P}$, we have

$$\hat{\gamma}_t = \sigma_t^* (\sigma_t \sigma_t^*)^{-1} [\mu_t - r_t 1 - \lambda_S], \qquad 0 \le t \le T \tag{44}$$

If $\tilde{\gamma} \in L_a^2[0, T]$ satisfies (44), then $\tilde{\gamma}$ can be written as[2]

$$\tilde{\gamma} = \hat{\gamma} + \vartheta \qquad \text{for some } \vartheta \in K(\sigma)$$

Indeed, decomposing $\tilde{\gamma}$ as $\tilde{\gamma} = \vartheta + \sigma^* \pi$ with $\vartheta \in K(\sigma)$yields by (43)

$$[\mu_t - (r_t + \lambda_B)1] = \sigma \tilde{\gamma} = \sigma \sigma^* \pi$$

Then the equation (43) becomes

$$\tilde{\gamma} = \vartheta + \sigma^* (\sigma \sigma^*)^{-1} [\mu_t - r_t 1 - \lambda_S]$$

This allows us to prove the following result.

Lemma: Every compatible asset has a value process A of the form

$$\begin{aligned} dA_t = &\left(\pi_t^* [\mu_t - r_t 1 - \lambda_{\pi^*}] + (r_t + \lambda_A) A_t\right) dt \\ &+ \pi_t^* \sigma_t dW_t + \nu_t^* dW_t + \nu_t^* \vartheta_t dt \end{aligned} \tag{45}$$

for some portfolio process π and some process $\nu, \vartheta \in K(\sigma)$.

proof: Let A and his equivalent A' a continuous processes. $\tilde{P}$ an equivalent martingale measure for S such that A' is a local $\tilde{P}$-martingale. Then $A'\tilde{P}$ is a local P-martingale and therefore continuous, since $\mathbb{F}$ is a Brownian filtration. We denote by $\tilde{\gamma}$ the process corresponding to $\tilde{P}$ by the relation (42). Under P, $A' = A/B$ has the form:

$$dA_t' = \frac{1}{B_t} dF_t - A_t'[r_t + \lambda_B] dt + \frac{\upsilon_t^*}{B_t} dW_t$$

where we have used equation. If we decompose (39)$\upsilon \in L_a^2[0, T]$ as:

$$\upsilon = \nu + \sigma_t^* \pi \qquad \text{with} \quad \nu \in K(\sigma)$$

then applying Girsanov's theorem to W shows that A' can be written under $\tilde{P}$ as

$$dA_t' = \frac{1}{B_t} dF_t - \left(A_t'[r_t + \lambda_B] + \frac{\upsilon_t^*}{B_t}\tilde{\gamma}_t\right) dt + \frac{\upsilon_t^*}{B_t} d\tilde{W}_t$$

for some $\tilde{P}$-Brownian motion $\tilde{W}_t$.

Since A' is a continuous local $\tilde{P}$-martingale, by substitution of $\tilde{\gamma} = \hat{\gamma} + \vartheta$ and using equation (44) in the previous equation we obtain:

$$\begin{aligned} dF_t &= \left(A_t[r_t + \lambda_A] + \upsilon_t^* \tilde{\gamma}_t\right) dt \\ &= \left(A_t[r_t + \lambda_{A_t}] + (\nu_t^* + \pi_t^* \sigma_t)(\hat{\gamma}_t + \vartheta_t)\right) dt \\ &= \left(A_t[r_t + \lambda_{A_t}] + \nu_t^* \hat{\gamma}_t + \nu_t^* \vartheta_t + \pi_t^* [\mu_t - r_t 1 - \lambda_{\pi^*}]\right) dt \end{aligned}$$

with F un $\mathbb{F}$-adapted process with paths of finite variation.

This equation confirm that the process of a general asset defined in equation is equivalent to the process defined in equation (45).

5 Conclusion

This paper developes a general context for the valuation of options with stochastic volatility and information costs. The shadow costs are integrated in the investor's portfolio wealth process in the same vein as in Merton (1987), Bellalah and Jacquillat (1995) and Bellalah (1999). The information costs appear naturally in the derivation proposed in this analysis. There is also another reformulation of the compatible asset's process, that gives more information for the drift term, which depends on information costs. In the same way, several extensions of existing models can be used for the development of the option valuation with stochastic volatility and information costs.

FOOTNOTES & REFERENCES

[1] see Norbert, Platen and Schweizer[1992] (page 162, 163)
[2] see Norbert, Platen and Schweizer[1992] (page 165)

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